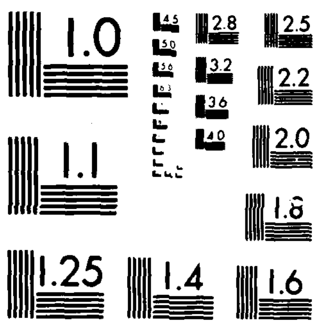
AD-A163 652 ON THE SHORE SINGULARITY OF WATER-WAVE THEORY PART 2 SMALL WAVES DON'T BR..(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER R E MEYER OCT 85 1/1

UNCLASSIFIED MRC-TSR-2873 DAAG29-80-C-0041 F/G 20/4 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

AD-A163 652

(2)

MRC Technical Summary Report # 2873

ON THE SHORE SINGULARITY OF WATER-WAVE THEORY. Part II. SMALL WAVES DON'T BREAK ON GENTLE BEACHES

R. E. Meyer

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

October 1985

(Received May 21, 1985)

Approved for public release
Distribution unlimited

Sponsored by

U.S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ON THE SHORE SINGULARITY OF WATER-WAVE THEORY. PART II. SMALL WAVES DON'T BREAK ON GENTLE BEACHES

R. E. Meyer

Technical Summary Report #2873

October 1985

ABSTRACT

The model of gravitational surface waves on beaches of small slope formulated in Part I[1] and its mathematical theory[2] are used to show how an incident-wave amplitude can be defined so that a bound on it guarantees solutions which respect the assumptions of the model everywhere and forever. The structure of those solutions "far" from shore is then compared with that predicted "near" shore by the classical, linear theory[3] to remove the indeterminacies of both theories: Shore reflection is determined for the classical theory, and it is shown how the critical length scale and amplitude of the beach theory are related to the familiar wavelength and amplitude in deep water. These results indicate that the beach theory~~[1,2,4-8]~~ captures and elucidates the basic singularity structure underlying the shore behavior of gravitational surface waves.

AMS (MOS) Subject Classifications: 76B15

Key Words: Waves on Beaches, Amplitude Bounds

Work Unit Number 2 - Physical Mathematics

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and partly supported by the National Science Foundation under Grant MCS-8215064 and a Senior Fellowship of the British Science and Engineering Research Council; the author is also indebted to the Imperial College, London, for hospitality during the drafting of the report.

SIGNIFICANCE AND EXPLANATION

The shore processes of oceanography are of decisive importance to the containment of beach erosion by the Corps of Engineers. They are also of interest to Geology in regard to continental accretion by sedimentation, and to Biology, in regard to the ecology of coastal wetlands and to the origins of life at the land-sea interface.

A question central to all those matters is how water waves behave on natural beaches close to shore, and progress in coastal oceanography has been long held up by a lack of theoretical understanding of that wave mechanism. Waves on beaches, of course, have attracted mathematical interest for a long time, and it has been recognized gradually that the shore singularities of the classical theories are an incorrect description of the shore mechanism.

The present work aims to make a convincing contribution to the identification and elucidation of the real singularity which embodies the underlying structure of water motion close to shore. It is an essentially nonlinear mechanism, but one very different from the nonlinearity of waves of large amplitude far from shores. It dominates waves of small amplitude as well, and this report concentrates on unbroken waves to complement earlier explorations of breaker collapse and conversion to run-up to a clear and balanced picture.

For unbroken waves, the new beach theory developed in the Reports TSR #'s 2871 and 2872 is especially successful because it will here be shown to overlap with the classical theory of small-amplitude waves. That produces a complete mathematical description of the inviscid structure underlying unbroken waves over natural beaches, which reaches from deep water to the moving shoreline. It encompasses standing waves as well as waves developing almost arbitrarily in time, and shows plainly how the nonlinear shore singularity, which manifests itself fully only close to the moving shoreline, nevertheless determines wave structure far into the deeper water.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE SHORE SINGULARITY OF WATER-WAVES THEORY.
PART II. SMALL WAVES DON'T BREAK ON GENTLE BEACHES

R. E. Meyer

I. Introduction

A model for gravitational surface waves close to shore over beaches of small slope was formulated in Part I[1] (hereafter referred to as [F.] with note of section or equation number). That formulation allows for the study of waves which "do not break", that is, of solution of the nonlinear beach equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(h + \frac{1}{2}u^2\right) = -1 \tag{2}$$

which satisfy the premises of the model [F. IV] in a strict sense. The notation and its physical connotations are explained at length in [F. IV, V]. The mathematical theory of the model is developed in a related report[2] (hereafter referred to as [M.]). The account to follow now concentrates on the oceanographical interpretation and application of the mathematical theory.

After a general characterization of the main class of model-preserving solutions in Sections II, III, the account turns to the task of removing a serious shortcoming of the theory: The general model has physical validity only in a close neighborhood of the moving shoreline, the extent of which it cannot determine. That is a fruitful restriction, on the one hand, because it directs the analysis toward the key to wave structure on beaches of small slope. It also recognizes that the model does not describe the observed[9] mechanisms of wave-breaking during shoaling. It is a great handicap, on the other hand, because it leaves unclear how the input to the model could be determined from what is known or observable of waves. A major objective now will be to show that this difficulty is not present for model-preserving solutions, on the contrary, the model describes for them the structure of most of the shoaling process.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and partly supported by the National Science Foundation under Grant MCS-8215064 and a Senior Fellowship of the British Science and Engineering Research Council; the author is also indebted to the Imperial College, London, for hospitality during the drafting of the report.

To explain this, it is advantageous to restrict attention in Sections IV, V to the special case of standing waves, for which the discussion can be concrete and explicit. There, the extension of the model-preserving, standing-wave solution beyond the model's a-priori domain of validity is compared with the extension of the solution[3] of the classical linear theory of small-amplitude waves on beaches to the smallest shore distances at wh ch that theory can describe water waves. The two solutions turn out to have the same fi (and close) approximation over an interval of distances if, and only if, a certain condition (Equation (17) below) is satisfied.

This condition removes the basic indeterminacies of both theories by determinin n the one hand, the correct linear combination of the fundamental solutions of the linea theory, and on the other hand, the unknown length scale L of the beach theory [F. IV] nd the beach amplitude (Section II) in terms of the deep-water wavelength and amplitude. A full description of the model-preserving standing waves from the deep sea to the moving shoreline is thereby established. It is used in Section V to derive also some simple and illuminating, if rough, quantitative predictions about the amplification of such waves by shoaling. Altogether, standing waves are simple and specific enough to display a good deal of physical flesh (Sections IV, V) on the powerful, but somewhat abstract, mathematical bones.

Section VI returns to the discussion of waves of much more general time-dependence. For model-preserving solutions, the "Friedrichs Connection" between beach theory and classical, linear theory is there shown to cover the case of quite general time-dependence to the extent that the waves have a Fourier transform with respect to time in deeper water. Between them, the two models offer a complete shoaling description also for such general waves developing in time. The condition for model-preservation (Section VI) is then mainly a restriction on the high-frequency part of the amplitude spectrum.

II. Amplitude

If a solution of the beach equations (1), (2) is to preserve their physical validity as an approximate description of water waves on beaches, it is necessary that the associated, apparent solution be invertible, as will be seen clearly in the next Section. For this, in turn, the Inversion Criterion [M. 5, F. X] states a necessary and sufficient condition for the class of incident waves specified by 3-compatible data:

$$\lim_{\sigma \to 0} \partial t/\partial \lambda$$

must have a positive lower bound. It is a rigorous, but not a practical condition. A useful one should be in terms of a relatively simple property of the incident wave. Of course, the Riemann representation established in the existence proof [M. 3] offers a means of representing $\lim \partial t/\partial \lambda$ as a functional of the data $\hat{a}(\alpha)$ specifying the incident wave, but the resulting, lengthy formulae are too complicated for practical use. Those data, moreover, specify the incident wave at an unknown, and possibly quite small, distance from shore. They need to be related to oceanographically meaningful properties at distances where the incident wave can be reasonably observed or generated.

In selecting a practical amplitude concept for the characterization of model-preserving solutions, it is important to understand the choices and limitations. An important limitation arises from the fact that, for a nonlinear, singular wave equation such as (1), (2), the amplitude concept depends on the solution class. It is different, for instance, for the highly singular solutions[4,5] related to "bore collapse" and for the less singular solutions[10,11] for (-1)-compatible data: A correct amplitude concept is essentially hinged to the fine print in its definition. The mathematical theory [M. 5] shows plainly that 1-compatibility is insufficient for global inversion of apparent solutions, and no general criterion for it has yet been found for 2-comparible data. At present, a reliable amplitude concept for model-preserving solutions must therefore be restricted to the class of 3-compatible incidence data. This is quite adequate for oceanographical purposes because it requires [M. 3] much less smoothness than waves are expected to preserve during shoaling short of breaking.

A related restriction is that the data must be consistent with local inversion on the incidence boundary. For that

$$\hat{a}\left(\frac{\sigma+\lambda}{2}\right) > -(\sigma + \lambda + 2\bar{\sigma})^{3/2}/2^{7/2} \tag{3}$$

is necessary and sufficient. Sufficient, because the Jacobian $J = \partial(x,t/\partial(\alpha,\beta)$ of the characteristic transformation [F. V] is

$$J = \frac{1}{2}\,\sigma[(\partial t/\partial\lambda)^2 - (\partial t/\partial\sigma)^2] \quad , \tag{4}$$

by [F. (4), (6), (12)], and by [F. (8), (12), (13), (20)], $\partial t/\partial\sigma = 0$, $\partial t/\partial\lambda = 1/2$ initially; by [F. (8), (12)], (3) leaves $\partial t/\partial\sigma + \partial t/\partial\lambda > 0$, and by [F. (8), (9), (12)], $\partial t/\partial\sigma - \partial t/\partial\lambda$ then remains negative along the incidence boundary $\sigma - \lambda = \text{const} = 2\bar{\sigma}$. The inequality (3) is also necessary because $\partial t/\partial\sigma + \partial t/\partial\lambda = \frac{1}{2} + 2\sigma^{-3/2}\,\hat{a}$ would otherwise have a root, and by [F. (8), (9), (12)], $\partial t/\partial\sigma - \partial t/\partial\lambda$ would still be negative at the first such root and it would be a root also of the Jacobian.

Among the choices is whether amplitude should refer to velocity or surface elevation or, as has come to be preferred often in oceanography, to an intrinsically nondimensional quantity such as surface slope $\partial\eta/\partial x$. The last commends itself by the intimate connection between $\partial\eta/\partial x$ and fluid acceleration documented by the momentum balance (2) and will be adopted here. That prompts, in turn, that amplitude be formulated as a property of the acceleration measure a characteristic of the incident wave [F. V] and in particular, of the function $\hat{a}(2\sigma-2\bar{\sigma})$ in [F. (22)] which specifies the incident wave in the mathematical beach theory [M. IX]. The simplest choice is then the following.

Definition. For 3-compatible incidence data $\hat{a}(\alpha) > -(\alpha+\bar{\sigma})^{3/2}/4$ on $[0,\bar{\alpha}]$, beach amplitude means

$$\max_{[0,\bar{\alpha}]} |\hat{a}(\alpha)| = \delta \quad .$$

III. Small waves never break

Inversion Theorem. An apparent solution is globally invertible if the beach amplitude δ is not too large.

This is a direct corollary [M. 6] of the Inversion Criterion because the canonical equations [F. (9), (10)] are linear and homogeneous, so that the characteristic accelerations a and b scale in proportion to δ throughout the apparent domain [F. (23)], and because [F. (13)] and the Regularity Corollary R.3a [M. 4] show $\partial t/\partial\lambda - \frac{1}{2} = \sigma^{-3/2}(a-b)$ to have a limit as $\sigma \to 0$, which must also scale with δ.

In the first place, the Inversion Theorem documents only the opportunity for a direct examination of the apparent solutions in terms of the beach equations (1), (2): the functions $x(\alpha,\beta)$ and $t(\alpha,\beta)$ obtained by quadratures from $Y(\sigma,\lambda)$ and $Z(\sigma,\lambda)$ via [F. (13), (9), (10), (12) and (6)] then furnish a regular, parametric representation of velocity

$$u(x,t) = \frac{1}{2}(\lambda+\bar{\sigma}) - t$$

and surface elevation

$$\eta(x,t) = h + x = \sigma^2/16 + x \quad .$$

Beyond this, however, the inversion condition is a mathematical code-word for a physical bound. By [F. (11), (12)] and (4), (2), the surface slope is

$$\begin{aligned} \partial\eta/\partial x &= 1 - \sigma(\partial t/\partial\lambda)/(4J) \\ &= -\partial u/\partial t - u\partial u/\partial x \quad . \end{aligned} \tag{5}$$

To find a root of the Jacobian J therefore means that the beach equations (1), (2) have already broken down as a physical model [F. IV]: invertibility is a necessary condition for preservation of the model. Small beach amplitude δ assures that this necessary condition remains satisfied permanently.

It is not a sufficient condition, however. The real issue must be whether fluid accelerations, surface slopes or surface curvatures occur which are quantitatively large enough to shake our confidence in the scaling [F. IV] that leads to the beach equations as an approximate description of water-wave properties. Physical effects ignored in the derivation of this model must then be considered, and they must determine the interpreta-

tion of the phrase "quantitatively large enough." Any sufficient condition for confidence in the model must therefore involve quantitative bounds on the surface slope.

Since the Invertibility Corollary [F. X] shows the threat to the model to be greatest near the shoreline, it is appropriate to look for surface-slope bounds there, and

$$\lim_{\sigma\to 0} \partial\eta/\partial x = \lim_{\sigma\to 0} [\sigma^{-3/2}Y/(\frac{1}{2} + \sigma^{-3/2}Y)]$$

$$= -\lim_{\sigma\to 0} (\partial u/\partial\lambda)/(\partial t/\partial\lambda) \quad ,$$

by (5), (4), [F. (12), (13)] and the Regularity Corollary R3 [M. IV]. The righthand limit exists, by the Inversion Criterion, and since Y scales in proportion to δ, the surface slope has a permanent bound even at the shoreline. Hence, if physical considerations indicate a tolerance limit for surface slope magnitude, then a positive bound on the amplitude δ exists which will guarantee that the tolerance limit is respected.

Again, the theory [M] permits that amplitude bound to be evaluated from the Riemann representation of the apparent solution, but a very general prediction of the bound is too complicated to promise much profit. It must also be borne in mind that the model of a beach of uniform slope is designed to serve conceptual clarification more than direct description of real circumstances. Moreover, amplitude bounds on the incident wave in the close vicinity of the shore covered by the theory, up to this point, are not of direct value to oceanography. For these reasons, attention will be directed now to the prediction of bounds in more specific circumstances of practical interest, and to the extension of such predictions to bounds on the incident wave in deeper water, which are of oceanographic value. To this end, it is of great help to consider first a very special example admitting a concrete elucidation of such issues, before the discussion returns in Section VI to a much wider subset of the very general class of wave motions encompassed by the mathematical theory.

IV. Standing Waves

Carrier and Greenspan's Theorm[12]. If the beach equations (1), (2) have an apparent solution with velocity simple-harmonic in characteristic time, its velocity u and surface elevation η must be

$$u = \sigma^{-1}\partial\phi/\partial\sigma, \qquad \eta = x+\sigma^2/16 = \frac{1}{4}\,\partial\phi/\partial\lambda - \frac{1}{2}u^2 \quad , \tag{6}$$

respectively, with $\lambda = 2t + 2u - \bar{\sigma}$, $\sigma = 4h^{1/2}$,

$$\phi = 8\delta\, J(\frac{\omega\sigma}{2})e^{-i\omega\lambda/2} \tag{7}$$

and constants $\bar{\sigma} > 0$, $\delta > 0$ and ω. For

$$|\omega^3\delta| < 1 \quad ,$$

moreover, this apparent solution does represent a strong solution $u(x,t)$, $h(x,t)$ of the beach equations (1), (2), which has a bound $|\omega^3\delta|/(1-|\omega^3\delta|)$ on the magnitude of surface slope and fluid acceleration.

Such a standing wave, of course, can be realized only asymptotically as $t \to \infty$, and the present theory (Section II, III) adds the comment that it can develop in time from the undisturbed state, if the incident wave amplitude remains small enough during the development, and that it must then develop, if the incident wave tends ultimately to the incident part of the standing wave.

The proof implicit in the original account[12] will help both to illustrate the application of the Inversion Criterion and to prepare the discussion to follow. In terms of the independent variables σ, λ the characteristic form [F. (6)] of the beach equations reads

$$\frac{\partial x}{\partial\sigma} = u\frac{\partial t}{\partial\sigma} - h^{1/2}\frac{\partial t}{\partial\lambda}, \qquad \frac{\partial x}{\partial\lambda} = -h^{1/2}\frac{\partial t}{\partial\sigma} + u\frac{\partial t}{\partial\lambda} \quad ,$$

and if functions $\partial\phi/\partial\sigma$ and $\partial\phi/\partial\lambda$ are defined in terms of u and x as in the theorem, the last two equations are equivalent to

$$\frac{\partial}{\partial\sigma}\frac{\partial\phi}{\partial\lambda} = \frac{\partial}{\partial\lambda}\frac{\partial\phi}{\partial\sigma}, \quad \frac{\partial}{\partial\sigma}(\sigma\frac{\partial\phi}{\partial\sigma}) = \sigma\frac{\partial^2\phi}{\partial\lambda^2} \quad .$$

If u is to be simple-harmonic in λ with frequency $\omega/2$, say, it follows from the last equation that

$$\phi(\sigma,\lambda) = Z_0(\omega\sigma/2)e^{-i\omega\lambda/2}$$

in terms of Bessel functions or order n,

$$Z_n(\frac{\omega\sigma}{2}) = 8\delta\, J_n(\frac{\omega\sigma}{2}) + 8\delta_2\, Y_n(\frac{\omega\sigma}{2}) \quad ,$$

and from the definition of $\partial\phi/\partial\sigma$, $\partial\phi/\partial\lambda$, that

$$u = -\frac{\omega}{2\sigma} Z_1(\frac{\omega\sigma}{2})e^{-i\omega\lambda/2} \quad ,$$

$$\eta + \frac{1}{2}u^2 = -\frac{i\omega}{8} Z_0(\frac{\omega\sigma}{2})e^{-i\omega\lambda/2} \quad ,$$

$$2t = \lambda + \bar{\sigma} - 2u = \lambda + \bar{\sigma} + \frac{\omega}{\sigma} Z_1(\frac{\omega\sigma}{2})e^{-i\omega\lambda/2} \quad .$$

If $\delta_2 \neq 0$, this apparent solution is not admissible [F. X] because clock-time t does not then exist at the shore $\sigma = 0$.

With $\delta_2 = 0$, however, the solution is admissible [F. X], and

$$\frac{\partial t}{\partial\lambda} = \frac{1}{2} - \frac{2i\delta\omega^2}{\sigma} J_1(\frac{\omega\sigma}{2})e^{-i\omega\lambda/2} \quad .$$

Since ϕ is then analytic in σ and λ, the incidence data are n-compatible for all $n > 0$, and the Inversion Criterion (Section II) becomes here

$$\lim_{\sigma\to 0} \partial t/\partial\lambda = \frac{1}{2} - \frac{1}{2} i\delta\omega^3 e^{-i\omega\lambda/2} > \delta_1$$

for some $\delta_1 > 0$, which is satisfied if, and only if, $|\omega^3\delta| < 1$. For this simple, apparent solution, it can also be checked readily that the Jacobian (4) has roots at positive values of σ if, and only if, $|\omega^3\delta| > 1$.

By (4) and (5), moreover, the surface slope is

$$\partial\eta/\partial x = 1 + \frac{1}{2}(\partial t/\partial\lambda)/[(\partial t/\partial\sigma)^2 - (\partial t/\partial\lambda)^2]$$

and from the standard properties of Bessel functions it is seen to have magnitude

$$|\partial\eta/\partial x| < \sup_{\lambda} \lim_{\sigma\to 0} |\partial\eta/\partial x| = |\omega^3\delta|/(1 - |\omega^3\delta|) \quad , \tag{8}$$

so that the bound on δ corresponding to any desired bound on $|\partial\eta/\partial x|$ is readily determined. (It will emerge in the next Section that there is no loss of generality in normalizing the non-dimensional frequency ω to 1 for waves of fixed frequency.)

Naturally, this standing-wave solution has many interesting properties[12], of which the most important, perhaps, is the asymptotic decay with increasing distance from shore. From the asymptotics of the Bessel functions as $4|\omega|h^{1/2} = |\omega|\sigma \to \infty$, the explicit formulae

just quoted are seen to imply, in turn,

$$u \sim -8\delta\left(\frac{\omega}{\pi}\right)^{1/2} \sigma^{-3/2} \cos\left(\frac{\omega\sigma}{2} - \frac{3\pi}{4}\right) e^{-i\omega\lambda/2} \quad ,$$

$$\eta \sim -2i\delta\left(\frac{\omega}{\pi}\right)^{1/2} \sigma^{-1/2} \cos\left(\frac{\omega\sigma}{2} - \frac{\pi}{4}\right) e^{-i\omega\lambda/2} \quad ,$$

$$\omega\lambda/2 \sim \omega t - \omega\bar{\sigma}/2 + 0(\sigma^{-3/2}) \quad ,$$

$$|x| = \sigma^2/16 + 0(\sigma^{-1/2}), \qquad \omega\sigma/2 = 2\omega|x|^{1/2} + 0(|x|^{-1/4}) \quad , \tag{9}$$

$$u \sim -\delta(\omega/\pi)^{1/2} |x|^{-3/4} \cos(2\omega|x|^{1/2} - 3\pi/4) e^{-i\omega t} \quad ,$$

$$\eta \sim -i\delta(\omega/\pi)^{1/2} |x|^{-1/4} \cos(2\omega|x|^{1/2} - \pi/4) e^{-i\omega t} \quad . \tag{10}$$

The motion far from shore (in units of the still unknown, critical length scale L) therefore becomes simple-harmonic also in clock-time and the amplitude of the surface elevation comes to *conform to the Green*'s law typical of linear longwave (or tidal) theory.

V. Friedrichs Connection

The beach equations (1), (2) are in terms of the nondimensional variables

$$\left.\begin{aligned} &x_B = x^*/L, \quad z_B = z^*/(\varepsilon L), \quad t_B = Ut^*/L \ , \\ &h_B = h^*/(\varepsilon L) = \sigma^2/16, \qquad \eta_B = h_B + x_B = \eta^*/(\varepsilon L) \ , \\ &\omega_B = \omega_* t^*/t_B = L\omega_*/U, \qquad U^2 = \varepsilon g L \end{aligned}\right\} \tag{11}$$

where stars denote the dimensional quantities, and the critical length scale L still remains undetermined. The seaward asymptotics of the beach theory describes the solution character towards somewhat deeper water at somewhat larger distances from shore in the limit $\omega_B\sigma/2 \to \infty$. When the local amplitude decreases with distance from shore, as for standing waves (Section IV), e.g., then $h_B \sim - x_B$ and the limit is also described by

$$\frac{\omega_B\sigma}{2} \sim 2\,\frac{\omega_* L}{U}\left|\frac{x^*}{L}\right|^{1/2} \to \infty \ . \tag{12}$$

In dimensional notation, the first approximation to the surface elevation (6) of the Carrier-Greenspan[12] standing wave is then

$$\eta^* \sim -i\delta\varepsilon L\left(\frac{2L}{\varepsilon\lambda_\infty}\right)^{1/2}\left|\frac{\varepsilon\lambda_\infty}{2\pi x^*}\right|^{1/4}\cos\left(2\left|\frac{2\pi x^*}{\varepsilon\lambda_\infty}\right|^{1/2} - \frac{\pi}{r}\right) e^{-i\omega_* t^*} \ , \tag{13}$$

by (9) - (11), where

$$\lambda_\infty = 2\pi g/\omega_*^2$$

is the dimensional "deep-sea" wavelength.

The solution of the classical, linear theory of irrotational, gravity waves over a beach of uniform, small slope ε has been given by Friedrichs[3]. His notation uses the variables

$$(x_F, z_F) = (2\pi/\lambda_\infty)(x^*, z^*), \qquad t_F = -\omega_* t^* \ ,$$

$$\eta_F = 2\pi\eta^*/\lambda_\infty \ ,$$

and like the beach theory, his "second asymptotic" description explores the limit $\varepsilon \to 0$ with εx^* fixed, but at a possibly different order of εx^*, namely, on the scale $\lambda_\infty/(2\pi)$. The regular standing wave of the classical, linear theory is then described by

$$\eta_F^R = 2i\delta_R A_1(\lambda)\cos\left(\frac{k(\lambda)}{\varepsilon} - \frac{\pi}{4}\right) e^{it_F} \tag{14}$$

in terms of certain functions[3] of the local wavelength λ and of a normalization factor δ_R.

The "deep-sea" asymptotics, as $\varepsilon x_F \to \infty$, of this description is given[3] by $A_1 \sim (2\pi\varepsilon)^{1/2}$, $k \sim \varepsilon x_F + \pi^2/8$ and therefore, by

$$\eta_F^R \sim -2i\delta_R(2\pi\varepsilon)^{1/2}\sin\left(x_F - \frac{\pi^2}{8\varepsilon} - \frac{\pi}{4}\right) e^{it_F} \quad \text{as} \quad \varepsilon x_F \to \infty \ ,$$

so that the "deep-sea" wavelength is just λ_∞ and the "deep-sea" amplitude,

$$\delta_\infty = (8\pi\varepsilon)^{1/2}\delta_R\, e^{i\pi} \ . \tag{15}$$

A diversion seems necessary here to clarify *the role of the* "deep-sea limit" in the theory of waves on beaches. It is not thought of as a realized wave state, but as the most natural and convenient reference state independent of the circumstances of a particular experiment or particular offshore topography. To the extent that the classical, linear theory, and common approximations[13] for naturally gentle seabed topographies can serve, a simple algorithm -- which is the more accurate[3], the smaller the beach slope -- relates this reference state to the local wave state at the position on a beach where the undisturbed depth is εx^*. *This is the sense* in which the deep-sea values of wavelength and amplitude for a beach of uniform slope were used as reference by Friedrichs[3] and will continue to be used here.

Friedrichs' account makes clear why the scaling underlying the classical, linear theory precludes its extension to the shore. The description (14), accordingly, applies only for $\varepsilon x_F > \text{const} > 0$, but there is no mathematical obstacle to the choice of a small number for this constant and to the exploration of (14) for values of εx_F near such a number. The *first approximation* to (14) for values of εx_F small in this sense is[3]

$$\begin{aligned}\eta_F^R &\sim 2i\delta_R\left(\frac{\pi^2\varepsilon}{x_F}\right)^{1/4}\cos\left(2\left(\frac{x_F}{\varepsilon}\right)^{1/2} - \frac{\pi}{4}\right) e^{it_F} \\ &= -i\delta_\infty(2\varepsilon)^{-1/2}\left|\frac{\varepsilon\lambda_\infty}{2\pi x^*}\right|^{1/4}\cos\left(2\left|\frac{2\pi x^*}{\varepsilon\lambda_\infty}\right|^{1/2} - \frac{\pi}{4}\right) e^{-i\omega_* t^*} \ ,\end{aligned} \tag{16}$$

which represents the same function $\eta^*(x^*,t^*) = \lambda_\infty \eta_F/(2\pi)$ as (13) if and only if,

$$4\pi(L/\lambda_\infty)^{3/2} = \delta_\infty/(\varepsilon\delta) \quad . \tag{17}$$

Together with the amplitude bound of Carrier's Theorem (Section IV), (17) therefore gives the necessary and sufficient condition for a coherent and rigorously founded approximation to the description of unbroken, gravitational, standing waves without vorticity over a plane beach from the deep sea right to the moving shoreline.

It is disappointing, but also illuminating, that (17) establishes only a relation between the scale L and the shore amplitude δ, given the reference wave farther from shore. The beach model is more subtle than could have been suspected at the beginning [F. IV], it involves an organic connection between length scale and amplitude.

This does not, however, damage the determinacy of the model in regard to observables. For instance, if amplitude is thought of in terms of energy, then it may be measured, far from shore, by the maximum of the surface elevation $|\eta^*|$ so that the deep-sea amplitude is $|\delta_\infty|\lambda_\infty/(2\pi)$; near shore, Carrier and Greenspan's Theorem (Section IV) shows the correct measure to be the maximum of $|\eta^*+u^{*2}/(2g)|$, which the theorem shows to be $\varepsilon L\omega_B\delta$, i.e.,

$$[\pi/(2\varepsilon)]^{1/2}$$

times the deep-sea amplitude, by (17). The amplification factor $\varepsilon^{-1/2}$ for beach slopes has been identified also in more heuristic descriptions[14]. Similarly, the identification of (13) with (16) by means of (17) demonstrates how knowledge of frequency and reference amplitude δ_∞ suffices to determine the detailed shore behavior of unbroken standing waves completely via Carrier and Greenspan's Theorem (Section IV).

An even more rational definition of amplitude is in terms of the non-dimensional surface slope $\partial\eta^*/\partial x^*$, which is also of prime dynamical relevance, by (2). In the deep sea, $\max|\partial\eta^*/\partial x^*|$ is just $|\delta_\infty|$, and at the shore, it is $\varepsilon|\omega_B^3\delta|/(1 - |\omega_B^3\delta|)$, by (9); and by (17),

$$\omega_B^3\delta = (\pi/2)^{1/2}\varepsilon^{-5/2}\,\delta_\infty \quad .$$

The amplification of surface slope is therefore $\varepsilon^{-3/2}/[(2/\pi)^{1/2} - \varepsilon^{-5/2}\delta_\infty]$, and the inversion condition $|\omega_B^3\delta| < 1$ necessary and sufficient for unbroken standing waves

translates into the deep-sea condition[8]

$$\delta_\infty < (2/\pi)^{1/2}\varepsilon^{5/2} \quad .$$

This expresses the true sense in which the amplitude must be small for waves that "never break" [F. II]: the amplitude far from shore must be small, not the shore amplitude δ. The precise condition just given is unrealistically stringent, however, because viscosity and surface tension have been ignored.

From a mathematical point of view, the identification of (13) and (16) represents only a "first-order match", and from an oceanographical point of view, it leaves us without a direct "feel" for the scales of the motion. Both these shortcomings are alleviated by numerical results[3] for the beach slope $\varepsilon = \pi/30$. Even for such a steep beach, there is close quantitative agreement[3] between the first-order approximation common to both theories and the exact, standing-wave solution of the classical, linear theory over a distance extending from about one-half to one deep-sea wavelength from shore. The length of this overlap interval corresponds to about one local wavelength, at such a large beach slope, and at the deep end of the interval, the water depth already exceeds 2/3 of the wave-penetration depth. The values of $\omega_B\sigma$ in the overlap interval are of order $1/\varepsilon$, and the first asymptotic approximation (Section IV) is then very close to the exact, standing-wave solution (Section IV) of the beach theory. In short, even at larger beach slopes than are common in nature, there is a substantial interval over which the exact solutions of the two theories differ by much less than could be detected by observation. Remarkably, moreover, the beach theory is seen thus to describe the whole shoaling process from the shoreline to a distance normally considered as one at which the water is "quite deep"; in particular, it covers all distances from shore over which the water would be considered shallow enough for application of the linear longwave theory.

For model-preserving solutions, the "confidence domain" of the beach equations (1), (2) is therefore very much larger than it can be for all solutions [F. II]. It extends to distances $\approx 10\varepsilon^{-1}\lambda_\infty$ from shore, at least, and the incidence condition [F. (22)] retains its direct, physical meaning over (scaled) times t of order ε^{-1}, at least; its mathematical extension [F. IX] is dictated only by the requirement of time-uniformity

[F. VIII] and by the objective of constructing a theory not at all limited to near-standing waves.

A different aspect of these issues is illuminated by the question: over what distance from shore do even "unbroken" waves display a seriously "nonlinear" character? To fix the ideas, if $2\pi^{1/2} \approx 3.5$ be regarded as the value below which the argument in the Bessel functions of Section IV "is not large", then since

$$(\omega_B \sigma/2)^2 = 4\omega_* h^*/(\varepsilon^2 g) = 8\pi|x^*|/(\varepsilon\lambda_\infty) \quad ,$$

asymptotic approximation of the Carrier-Greenspan standing wave fails only for

$$|x_B| < \pi, \qquad |x^*| < \varepsilon\lambda_\infty/2 \quad .$$

That the nonlinear beach theory is needed for the <u>local</u> description of waves over only such tiny distances from shore must not, of course, be permitted to obscure its qualitative long-range influence: it is the theorem of Section III which makes the selection between the two fundamental solutions of the linear theory.

The critical length scale L is defined [F. IV] as the scale of distances from shore over which the nonlinear wave mechanism dominates. As a scale, of course, it cannot be assigned a unique numerical value, but any reasonable estimate of it will help greatly to fix the ideas. If $\omega_B \sigma/2 = 2\pi^{1/2}$ be again chosen to correspond to the distance from shore at which the seaward asymptotics of the beach theory becomes applicable and hence, the non-linear aspect of that theory has ceased to dominate, then the distance x^* at which $\omega_B \sigma/2 = 2$ appears a reasonable choice of the critical scale L for illustrative purposes. It corresponds to the choice $L = U/\omega_*$, i.e., $\omega_B = 1$. Its dimensional value is

$$L = \varepsilon\lambda_\infty/(2\pi) \quad .$$

To illustrate the striking implications of this long-delayed insight, the Table lists these values of L for various beach slopes ε and reference wavelengths λ_∞.

Table

	$\varepsilon = 10^{-1}$	10^{-2}	10^{-3}
$\lambda_\infty =$ 10 cm	L = 1.6 mm		
1 m	1.6 cm	1.6 mm	0.16 mm
10 m	16 cm	1.6 cm	1.6 mm
100 m		16 cm	1.6 cm

The scale L is much smaller than intuition could have suggested, and this emphasizes again the dual role of the shore singularity: the singular enhancement of the shoaling process becomes pronounced only in the immediate neighborhood of the moving shoreline, but the beach solutions describing it also describe the wave motion over the beach at much larger distances and have a decisive influence on the wave structure in the deeper sea.

The small values of L in the table suggest that surface tension should play an important role near the shoreline, but surface curvature there depends on the local amplitude even more than on the length scale, and the issue is beyond the scope of the present study.

VI. Second Friedrichs Connection

To extend such considerations of a match between shore theory and linear theory on a firm basis to more general, developing waves, requires a useful representation of such solutions of the beach equations (1), (2). A very general, Riemann representation is described in the proofs (Appendices A, B of [M]) of the Incidence and Regularity Theorems, but that representation is complicated due, not least, to its very generality. Waves of oceanographical interest incident from the sea would be expected to be quite smooth, and the Incidence Corollaries [M. 4] show that such smoothness carries over to the apparent solutions. Such waves can therefore be expected to possess a Fourier transform with respect to characteristic time λ, if they decay sufficiently as $\lambda \to \infty$.

To derive the Fourier representation from incidence data is complicated and can be avoided by recalling the uniqueness of apparent solutions [M. 3]. If σ_1 is a small, positive number, the apparent problem is regular for $\sigma > \sigma_1$, and in Carrier's notation (Section IV), the solution of the wave equation for his function $\phi(\sigma,\lambda)$ is determined uniquely for $\sigma > \sigma_1$ by the Cauchy data $\{\phi(\sigma_1,\lambda),\ \sigma_1 u(\sigma_1,\lambda)\}$ together with the initial data. If the shore conditions [F. (18), (19)] are to be satisfied as well, however, the functions $\phi(\sigma_1,\lambda)$ and $\sigma_1 u(\sigma_1,\lambda)$ must depend on each other, and thus $\phi(\sigma_1,\lambda)$ already determines any admissible solution uniquely.

The class of apparent solutions under consideration now can therefore be characterized by the assumption that $\phi(\sigma_1,\lambda)$ has a Fourier transform

$$8\delta_B(\omega)\ J_0\left(\frac{\omega\sigma_1}{2}\right) = \frac{1}{2}\int_{-\sigma_1}^{\infty} \phi(\sigma_1,\lambda)e^{i\omega\lambda/2}d\lambda$$

(since $\phi(\sigma_1,\lambda) \equiv 0$ for $\sigma_1 + \lambda < 0$ for the solutions starting from rest) in the strict sense, if $\phi(\sigma_1,\lambda)$ decays sufficiently as $\lambda \to \infty$, and in the distributional sense, in any case. The non-dimensional frequency is here again $\omega = L\omega_*/U$ in terms of the dimensional frequency ω_*. The uniqueness of admissible solutions then implies that it must have the Carrier function

$$\phi(\sigma,\lambda) = \frac{4}{\pi}\int_{-\infty}^{\infty} \delta_B(\omega)J_0\left(\frac{\omega\sigma}{2}\right)e^{-i\omega\lambda/2}d\omega$$

because this solves the apparent equation and has the correct Fourier transform of

$\phi(\sigma_1,\lambda)$. This confirms the plausible expectation that any beach solution of amplitude small enough to guarantee model-preservation can be represented as a superposition of standing waves.

It now follows from the initial definition of ϕ (Section IV) that

$$\eta_B + \frac{1}{2}u^2 = \frac{1}{4}\frac{\partial\phi}{\partial\lambda} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \omega\delta_B(\omega)J_0(\frac{\omega\sigma}{2})e^{-i\omega\lambda/2}d\omega$$

and

$$u = \frac{1}{\sigma}\frac{\partial\phi}{\partial\sigma} = \frac{-2}{\pi\sigma}\int_{-\infty}^{\infty} \omega\delta_B\, J_1(\frac{\omega\sigma}{2})e^{-i\omega\lambda/2}d\omega \quad ,$$

which decays even faster as $\sigma \to \infty$ at fixed λ, so that the Fourier transform of η_B becomes ultimately

$$\hat{\eta}_B \sim -i\omega\delta_B(\omega)\; J_0(\omega\sigma/2)$$

$$\sim -2i(\omega/\pi\sigma)^{1/2}\,\delta_B(\omega)\;\cos\{\omega\sigma/2 - \pi/4\} \quad \text{as} \quad |\omega\sigma| \to \infty \quad ,$$

and from [F. (12)], also $\lambda/2 \sim t_B + \text{const}$. Since η_B decays, moreover, $\sigma^2/16 \sim -x_B$, and on recalling the scaling (11) of the beach theory, $\omega\sigma/2 \sim 2\omega_*|Lx^*|^{1/2}/U$, and the Fourier transform of η_B is approximated by

$$\hat{\eta}_B \sim -i\omega_*\delta_B L\;(\pi\omega_* U)^{-1/2}\,|Lx^*|^{-1/4}\cos\{2\omega_*|Lx^*|^{1/2}/U - \pi/4\} \tag{18}$$

as $|\omega\sigma| \to \infty$, always provided, of course, that the incident-wave amplitude is small enough.

To see what this proviso means in the present context, note that

$$\frac{\partial u}{\partial\lambda} = \frac{i}{\pi\sigma}\int_{-\infty}^{\infty} \omega^2\delta_B(\omega)J_1(\frac{\omega\sigma}{2})d\omega$$

and therefore from [F. (12)],

$$\frac{\partial t}{\partial\lambda} = \frac{1}{2} - \frac{\partial u}{\partial\lambda} = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} [1 - \frac{2\omega^3\delta_B}{\sigma}J_1(\frac{\omega\sigma}{2})]e^{-i\omega\lambda/2}\frac{d\omega}{\omega} \quad .$$

Since the righthand side here is $1/2$ for $\delta_B(\omega) \equiv 0$, it has a positive lower bound at the singular line $\sigma = 0$, if $\omega^4\delta_B(\omega)$ has a sufficiently small, upper bound.

The more general wave motions now under consideration possess no obvious amplitude constant and accordingly, also no obvious definition of a reflection coefficient. However, for waves of shore amplitude spectrum small enough to preclude "breaking", the inviscid beach model (1), (2) conserves energy and therefore implies reflected waves of amplitude

comparable to that of the incident waves. The waves further away from shore can accordingly be expected to be governed by the linear theory to a satisfactory approximation. To make that theory mathematically determinate requires recourse also to initial conditions, which are not expected, however, to be very relevant to what one wishes to observe in practice. It will therefore be more helpful to assume here that the wave motion far from shore possesses a frequency scale Ω characteristic of its spectrum, which can be used to make the governing equations non-dimensional. A non-dimensional time t_L, frequency ω_L, distance x_L, surface elevation η_L and velocity potential Φ_L may then be introduced by

$$t_L = \Omega t^*, \qquad \omega_L = \omega_*/\Omega, \qquad x_L = -x^*\Omega^2/g \quad ,$$

$$\eta_L = \eta^*\Omega^2/g, \qquad \Phi_L = \Phi^*\Omega^3/g^2 \quad .$$

In view of the expected smoothness of the waves of small amplitude, it is plausible to assume that Φ_L also possesses a Fourier transform

$$\hat{\Phi}(x_L,y_L;\omega_L) = \int_{-\infty}^{\infty} \Phi_L(x_L,y_L,t_L)\, e^{i\omega t}\, dt_L$$

in the same conditional sense appealed to for the solution of the beach equations. For every fixed ω_L, $\hat{\Phi}$ must then satisfy the equations solved by Friedrichs[3] and hence,

$$\hat{\Phi}(x_L,y_L;\omega_L) = \delta_+(\omega_L)\chi_+(z_F) + \delta_-(\omega_L)\chi_-(z_F)$$

in his notation[3], where $z_F = x_F + iy_F$ is referred to the deep-sea wave number corresponding to ω_L, i.e.,

$$z_F = \omega_L^2\Omega^2 z^*/g = \omega_L^2 z_L \quad .$$

The incident-wave components of the spectrum are then[3]

$$\delta_+\chi_+e^{i\omega t} + \delta_-\chi_-e^{-i\omega t} \sim 2C\delta_-\exp[-i(z_F-\omega t)]$$

in the deep-sea, with

$$C = -(2\pi\varepsilon)^{1/2}\exp[-i\pi^2/(8\varepsilon) + i\pi/4] \quad ,$$

so that

$$\delta_\infty(\omega_L) = 2C\ \delta_-(\omega_L)$$

represents the deep-sea amplitude spectrum of the incident waves, and of the reflected waves, as well, because[3] $\chi_+ \to 0$ in the deep sea.

The surface elevation predicted by the linear theory is

$$\eta_L = \lim_{y_L \to 0} \partial\Phi_L/\partial t_L \quad ,$$

so that its Fourier transform is

$$\hat{\eta}_L = i\omega_L \hat{\Phi}(x_L, 0; \omega_L) = \delta_+ \chi_+(x_F) + \delta_- \chi_-(x_F) \quad ,$$

and the approximation to it as the beach slope $\varepsilon \to 0$ for fixed $\varepsilon x_F = \varepsilon\omega_L^2 x_L$ is given by the functions $\chi_{+,2}$ and $\chi_{-,2}$ described by Friedrichs[3]. For $y_L = 0$ and small values of εx_F, those are approximated, in turn, by[3]

$$\chi_{\pm,2}(x_F) \sim -(\pi/\omega_L)^{1/2} (\varepsilon/x_L)^{1/4} \exp[\pm\, i(\nu+\pi/2)] \quad ,$$

$$\nu = 2\omega_L (x_L/\varepsilon)^{1/2} - \pi/4 \quad ,$$

and in that approximation

$$\hat{\eta}_L \sim (\pi\omega_L)^{1/2} (\varepsilon/x_L)^{1/4} \{\delta_+ e^{i\nu} - \delta_- e^{-i\nu}\} \quad . \tag{19}$$

When a comparison of the transforms of the actual surface elevations $\eta^* = \varepsilon L\eta_B$ and $\eta^* = g\eta_L/\Omega^2$ predicted by the two theories is now envisaged, it must be recognized that, while the frequency ratio $\omega_L/\omega = U/(L\Omega)$ is fixed, the frequencies themselves range over all the real numbers, and different spectral components will come to be compared at different shore-distances x^*. This is not meant in the sense that the real beach has uniform slope extending to infinite water depth and that the real motion remains two-dimensional over such infinite distances from shore. Both theories admit a mathematical extrapolation of local conditions to theoretical conditions at other locations over such an idealized beach, and it is this extrapolation which provides the conceptual clarification on which a reasonably simple correlation of the theories can be based. In reference to a plane beach of indefinite extent, moreover, a comparison of different spectral components at different shore distances is natural because any given $x^* \neq 0$ corresponds already to effective deep-sea conditions for the shortwave components, but still to effective shore-proximity, for the longwave components. Given any ω_L, however, it was seen in Section V that there is a common range of x^* over which (18) and (19) approximate the solutions of the respective theories.

With this understanding, a comparison of the approximation for $\varepsilon L\hat{\eta}_B$ given by (18) with that for $g\hat{\eta}_L/\Omega^2$ given by (19) becomes meaningful since the transforms are with respect to the same non-dimensional time in the limits envisaged. In the arguments of the trigonometric terms, $\omega_*^2 L|x^*|/U^2$ must be compared with $\omega_L^2 x_L/\varepsilon$, and they are identical because $\omega_L = \omega_*/\Omega$ and $U^2 = \varepsilon g L$, by definition. The trigonometric terms in (18) and (19) are therefore identical, if and only if

$$\delta_+(\omega_L) \equiv -\delta_-(\omega_L) \quad ,$$

by which the comparison is seen again to resolve the indeterminacy in the linear theory. The identification of $\varepsilon L\hat{\eta}_B$ and $g\hat{\eta}_L/\Omega^2$ is then complete, if and only if

$$- i\varepsilon\omega_*\delta_B L^2(\pi\omega_* U)^{-1/2} L^{-1/4} \equiv -2\delta_- g\Omega^{-2}(\pi\omega_L)^{1/2}(\varepsilon g/\Omega^2)^{1/4} \quad ,$$

i.e.,

$$\delta_B(\omega) \equiv -2\pi i\delta_-(\omega_L)U^2/(\varepsilon L\Omega)^2$$

$$= (\tfrac{\pi}{2})^{1/2}\delta_\infty(\omega_L)\varepsilon^{-3/2}\frac{g}{L\Omega^2}\exp[\frac{i\pi}{4}(1+\frac{\pi}{2\varepsilon})]$$

with $\omega = \omega_L\Omega(L/\varepsilon g)^{1/2}$. This determines the Fourier transform of the dimensional solution of the beach equations uniquely from the Fourier transform of the developing wave in the deep sea.

The relation between the amplitude spectra $\delta_B(\omega)$ and $\delta_\infty(\omega_L)$ given by the last equation still depends on the choice of the scales L and Ω, which are left somewhat indefinite by the mere specifications that Ω is a scale characteristic of the deep-sea frequency spectrum and L, a scale characteristic of the shore distance within which "nonlinearity" comes to dominate. In a consistent description, the two scales cannot be completely independent, and the results of Section V show that L must be envisaged to be of the order of $\varepsilon g/\Omega^2$. As seen at the end of Section V, moreover, the choice $L = \varepsilon g/\Omega^2$ is not unreasonable for fixing the ideas; it makes $\omega = \omega_L$ and simplifies the amplitude relation to

$$\delta_B(\omega_L)/\delta_\infty(\omega_L) = (\pi/2)^{1/2}\varepsilon^{-5/2}\exp[\frac{i\pi}{4}(1+\frac{\pi}{2\varepsilon})] \quad .$$

Except, perhaps, at the longwave end of the wave spectrum, the very large size of this ratio is an unrealistic feature arising from the neglect of dissipation in the present,

inviscid theory, which aims only to clarify the basic, nonlinear structure underlying waves on beaches.

REFERENCES

1. R. E. Meyer, On the shore singularity of water-wave theory, Part I, The local model MRC Tech. Sum. Rep. #2872.
2. R. E. Meyer, Regularity for a singular conservation law, MRC Tech. Sum. Rep. #2871.
3. K. O. Friedrichs, Water waves on a shallow sloping beach, Commun. Pure Appl. Math. 1, 109-134 (1948).
4. D. V. Ho and R. E. Meyer, Climb of a bore on a beach. Pt. I, Uniform beach slope, J. Fluid Mech. 14, 305-318 (1962).
5. M. C. Shen and R. E. Meyer, Climb of a bore on a beach, Pt. III, Run-up, J. Fluid Mech. 16, 113-125 (1963).
6. R. E. Meyer and A. D. Taylor, On the equations of surf, J. Geophys. Res. 68, 219-230 (1963).
7. R. E. Meyer, An asymptotic method for a singular hyperbolic equation, Arch. Rat. Mech. Anal. 22, 185-200 (1966).
8. R. E. Meyer and A. D. Taylor, Run-up on beaches, in Waves on Beaches, (R. E. Meyer, ed.) 357-412 (Academic Press, New York, 1972).
9. C. J. Galvin, Wave breaking in shallow water, ibid. 413-456.
10. R. E. Meyer, The method of characteristics for problems of compressible flow involving two independent variables, Pt. II. The radial focusing effect in axially symmetrical supersonic flow, Quart. J. Mech. Appl. Math. 1, 196-216 (1948).
11. Y. W. Chen, Supersonic flow through nozzles with rotational symmetry, Commun. Pure Appl. Math. 5, 57-87 (1952).
12. G. F. Carrier and H. P. Greenspan, Water waves of finite amplitude on a sloping beach, J. Fluid Mech. 4, 97-109 (1958).
13. R. E. Meyer, Theory of water-wave refraction, Adv. Appl. Mech. 19, 53-141 (1979).
14. J. B. Keller, Tsunamis -- water waves produced by earth quakes, Tsunami Hydrodynamics Conference 1961, IUGG monograph no. 24, 154-166 (1961).

REM/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
#2873

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
On the Shore Singularity of Water-Wave Theory. Part II. Small Waves Don't Break on Gentle Beaches

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
R. E. Meyer

8. CONTRACT OR GRANT NUMBER(s)
MCS-8215064
DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53705

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Work Unit Number 2 - Physical Mathematics

11. CONTROLLING OFFICE NAME AND ADDRESS
See Item 18 below

12. REPORT DATE
October 1985

13. NUMBER OF PAGES
22

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Waves on Beaches, Amplitude Bounds

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
The model of gravitational surface waves on beaches of small slope formulated in Park I and its mathematical theory are used to show how an incident-wave amplitude can be defined so that a bound on its guarantees solutions which respect the assumptions of the model everywhere and forever. The structure of those solutions "far" from shore is then compared with that predicted "near" shore by the classical, linear theory to remove the indeterminacies of both theories: Shore reflection is determined for the classical theory, and it is shown how the critical length scale and amplitude of the beach theory are related

(continued)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT (continued)

to the familiar wavelength and amplitude in deep water. These results indicate that the beach theory captures and elucidates the basic singularity structure underlying the shore behavior of gravitational surface waves.

END

FILMED

3-86

DTIC